The Crossing Number of Graphs: Theory and Computation

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Abstract. This survey concentrates on selected theoretical and computational aspects of the crossing number of graphs. Starting with its introduction by Turán, we will discuss known results for complete and complete bipartite graphs. Then we will focus on some historical confusion on the crossing number that has been brought up by Pach and Tóth as well as Székely. A connection to computational geometry is made in the section on the geometric version, namely the rectilinear crossing number. We will also mention some applications of the crossing number to geometrical problems. This review ends with recent results on approximation and exact computations.

1 Introduction

The crossing number $cr(G)$ of a graph is the smallest number of edge crossings achievable when laying out G in the 2-dimensional plane. The problem originated from Turán in 1944 when he worked in a labor camp [31]:

"There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there ... the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short, this caused a lot of trouble and loss of time ... the idea occured to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized."

In 1952, Turán mentioned this problem to Zarankiewicz, who presented a solution for the crossing number of complete bipartite graphs in 1954 [34]. Unfortunately, Ringel found a gap in the published proof that has not been closed yet.

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1.1 The Crossing Number for Complete Bipartite Graphs

However, the formula presented by Zarankiewicz is conjectured to be correct. His construction provides drawings with exactly

$$Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

crossings. Figure 1 shows the construction for $K_{6,6}$.

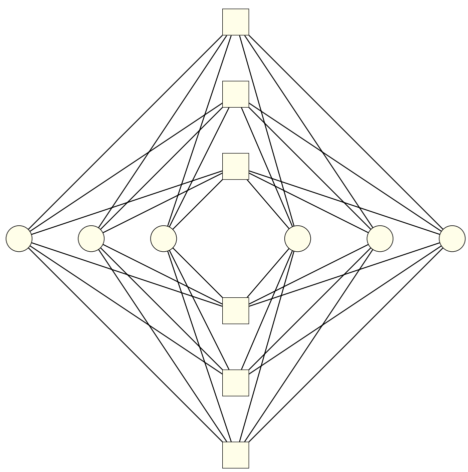

Fig. 1. Zarankiewicz's construction for $K_{6,6}$

The conjecture is known to be true for $K_{m,n}$ with $\min(m,n) \leq 6$ and for $m,n \leq 7$. The smallest unsolved cases are $K_{7,11}$ and $K_{9,9}$ with conjectured values 225 and 256, respectively. New bounding techniques using semi-definite programming [12] have shown that

$$0.8594 Z(m,n) \leq cr(K_{m,n}) \leq Z(m,n).$$

1.2 The Crossing Number for Complete Graphs

The crossing number for the complete graph K_n is not known either. It is generally believed to be given by the formula provided by Guy [18]:

$$Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor .$$

For odd n the formula can be written as $Z(n) = \frac{1}{64}(n-1)^2(n-3)^2$. Also Guy presented a general drawing scheme for K_n that produces drawings with exactly $Z(n)$ crossings. Guy's construction for K_8 is shown in Figure 2.

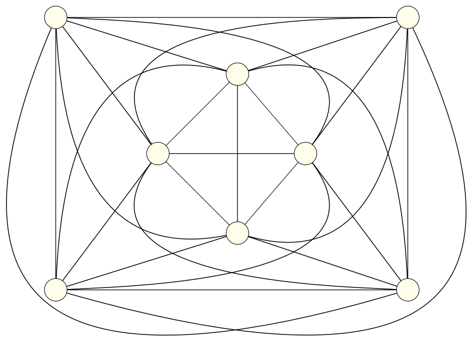

Fig. 2. Guy's construction for K_8

Combinatorial arguments show the following fact: If Guy's conjecture is true for K_{2k-1}, then it is also true for K_{2k}. This is the reason why the proofs concentrate on K_n for odd n. Guy [18] was able to prove his conjecture for K_n with $n \leq 10$. For 35 years, this result could not be improved. Recently, Pan and Richter [24] showed that $cr(K_{11}) = 100$ thus getting $cr(K_{12}) = 150$ for free. The smallest unsolved case is K_{13} with conjectured crossing number 225.

For many graph classes the situation is similar: $cr(G)$ is known for small instances only, while for general n not much is known (e.g., hypercube graphs, toroidal graphs, or generalized Peterson graphs). For a bibliography on the crossing number, see [33].

2 Confusion on the Crossing Number

In their interesting article [23] "What crossing number is it anyway?" Pach and Toth stated that "... some authors might have thought of ..." different crossing numbers. They pointed out that the definitions for the crossing number provided in the literature were not always the same. In order to investigate this further, we will use the formal definitions of Székely [30].

A *drawing* D of a graph G into the plane is an injection from the vertex set $V(G)$ into the plane, and a mapping Φ of the edge set $E(G)$ into the set of simple planar curves, such that the curve corresponding to the edge has end points $\Phi(u)$ and $\Phi(v)$, and contains no other vertices. The *number of crossings* $cr(D)$ in D is the number of intersection points of all unordered pairs of interiors of edges. The *crossing number* $cr(G)$ of a graph G is the minimum $cr(D)$ over all drawings D of G. A drawing D is *optimal* if it realizes $cr(D) = cr(G)$.

A drawing D is called *normal* if it satisfies

i any two of the curves have finitely many points in common
ii no two curves have a point in common in a tangential way
iii no three curves cross each other in the same point

A drawing is *nice*, if it is normal, and satisfies

iv no two adjacent edges cross
v any two edges cross at most once

It can be observed that an optimal drawing must satisfy i, ii, iv, and v, and can be transformed to satisfy iii. Therefore, we can restrict ourselves to consider normal or even nice drawings when interested in $cr(G)$.

Pach and Tóth [23] introduced two variants of crossing numbers: The *pairwise crossing number* $cr\text{-}pair(G)$ is the minimum number of edge pairs that cross each other at least once, over all normal drawings of G. The *odd crossing number* $cr\text{-}odd(G)$ is equal to the minimum number of edge pairs that cross each other an odd number of times, over all normal drawings of G.

In [32], Tutte introduces yet another version that Szégely calls the *independent odd crossing number* $cr\text{-}iodd(G)$. It is equal to the minimum number of non-adjacent edge pairs that cross each other odd times, over all normal drawings of G. The reason why Tutte introduced this crossing number was his "... view that crossings of adjacent edges are trivial, and easily get rid of." But so far nobody has shown that this can be done in this setting.

The following relation between these variants is obvious:

$$cr\text{-}iodd(G) \leq cr\text{-}odd(G) \leq cr\text{-}pair(G) \leq cr(G)$$

Pach and Tóth mention that "... perhaps the most exciting open problem in the area ..." is the question: "Are they all equal?"

One of the reasons why researchers thought that these numbers might be equal is an old theorem by Hanani [19], rediscovered by Tutte [32]. It states that every graph that can be drawn such that every pair of non-adjacent edges intersects an even number of times, is planar. Pach and Tóth [23] have generalized this result by showing that one can redraw even edges without crossings even in the presence of odd edges. Pelsmajer, Schaefer, and Štefankovič [25] have shown that this redrawing can be performed without adding pairs of edges that intersect an odd number of times; in particular, the odd crossing number does not increase by the redrawing. It is known that $cr(G) \leq 2(cr\text{-}odd(G))^2$ [23] and that for graphs with $cr\text{-}odd(G) \leq 3$ indeed $cr\text{-}odd(G) = cr(G)$ [25].

Some authors have stated the conjecture that $cr\text{-}odd(G) = cr(G)$. A surprising result by Pelsmajer, Schaefer, and Štefankovič [26] showed that equality of both crossing number variants does not hold. The authors have presented a quite simple infinite family of graphs with $cr\text{-}odd(G) < cr\text{-}pair(G) = cr(G)$.

Figure 3 shows an example of such a graph G. The four distinguished edges a, b, c and d have weights $w_a = 1$, $w_b = w_c = 3$, and $w_d = 4$. We assume that the weights of the edges e along the two main cycles are heavy so that they are not crossed in an optimal drawing (e.g., $w_e = 15$). We can think of replacing an edge with weight w by w parallel edges. It is also possible to get rid of parallel edges by subdividing these edges. The left drawing shows an optimal drawing for $cr(G)$ and $cr\text{-}pair(G)$. The crossing number and the pairwise crossing number is $cr(G) = cr\text{-}pair(G) = 15$. In the right drawing, the edges a and c cross

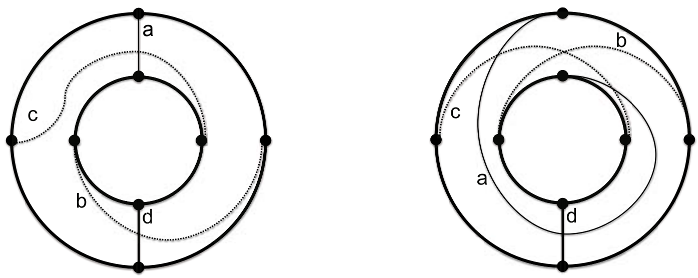

Fig. 3. A graph with $cr\text{-}odd(G) < cr(G)$. The left drawing shows an optimal drawing for $cr(G)$ and $cr\text{-}pair(g)$ with value 15. The right drawing shows an optimal drawing for $cr\text{-}odd(G)$ with value 13.

each other exactly twice providing the amount of 0 to $cr\text{-}odd(G)$ thus giving $cr\text{-}odd(G) = 13 < cr(G)$.

However, the question if $cr(G) = cr\text{-}pair(G)$ is still open. The bound of $cr(G) = O(cr\text{-}pair(G)^2/\log(cr\text{-}pair(G)))$ has been provided by Valtr (mentioned in [22]).

Now that we know that the four crossing numbers are not equal, a question already stated by Szégely is catching our interest [30]: "How is it possible that decades in research of crossing numbers passed by and no major confusion resulted from these foundational problems?" — Perhaps the graph theory community was just lucky that the bounds they provided in all these years apply for all kinds of crossing numbers.

3 The Rectilinear Crossing Number

The geometric version of the crossing number is called the *rectilinear crossing number*, denoted by *cr-lin*, and requires a drawing in the plane with straight line segments. It is well known that a planar graph always has a planar drawing with straight line segments. Therefore, one may think that $cr\text{-}lin(G)$ and $cr(G)$ are equal or close together. However, already Guy [17] has shown that $cr(K_8) = 18$ while $cr\text{-}lin(K_8) = 19$. Later, Bienstock and Dean [5] have shown that the two numbers are equal ($cr\text{-}lin(G) = cr(G)$) for graphs with small crossing number $cr(G) \leq 3$. On the other hand, the authors have also shown that there are graphs with crossing number 4 and arbitrarily large rectilinear crossing number. However, for graphs with bounded degree, the crossing number and the rectilinear crossing number are bounded as functions of one another [4]. In detail, if a graph has maximum degree d and crossing number k, its rectilinear crossing number is at most $O(dk^2)$.

It is conjectured that the construction by Zarankiewicz for the crossing number of complete bipartite graphs provides the correct numbers for $cr\text{-}lin(K_{m,n})$. Due to the nature of the construction, a proof for $cr(K_{m,n}) = Z(m,n)$ would directly lead to $cr\text{-}lin(K_{m,n}) = Z(m,n)$.

Until 2001, the rectilinear crossing number for the class of complete graphs K_n was known only for $n \leq 9$. Then, two groups of researchers independently showed that $cr\text{-}lin(K_{10}) = 62$. Aichholzer, Aurenhammer and Krasser [1] exhaustively enumerated all combinatorial inequivalent point sets (so-called *order types*) of size 10. Similar methods have been successful for showing $cr\text{-}lin(K_{11}) = 102$ and $cr\text{-}lin(K_{12}) = 153$. The authors initiated the *Rectilinear Crossing Number Project* [27] in which users provide their own computing power to the project. The main goal of the current project is to use sophisticated mathematical methods (abstract extension of order types) to determine the rectilinear crossing number for small values of n, and to compute all existing combinatorial non-isomorphic minimal drawings. Currently, the rectilinear crossing number is known for all K_n with $n \leq 21$ ($cr\text{-}lin(K_{21}) = 2055$). In contrast to $cr(K_n)$ there is no conjecture following some formula for arbitrarily large n for $cr\text{-}lin(K_n)$. However, the gaps between the lower and upper bounds for n up to 100 are quite small, e.g., $1.459.912 \leq cr\text{-}lin(K_{100}) \leq 1.463.970$.

4 Applications of the Crossing Number

Székely has used bounds for the crossing number $cr(G)$ for providing a simple proof of the Szemerédi-Trotter theorem, that is an important result in combinatorial geometry. It asks for the maximum number of incidences of n points and m curves in the plane such that each pair of curves intersects at maximal $O(1)$ points, and there are no more than $O(1)$ curves passing through each pair of points. The answer in this case is $O(m + n + (mn)^{2/3})$. The idea of Szégely's proof was to build a graph in which the vertex set is associated with the point set and the edges with the curve segments. Using bounds for the crossing number for complete graphs essentially provided the solution.

Szemerédi-Trotter like theorems can be used for proving hard Erdős problems in combinatorics, in number theory, in analysis or geometric measure theory. E.g., Elekes [14] used it to show that any n distinct real numbers have $\Omega(n^{1.25})$ distinct sums or products. And a famous problem of Erdős in geometry asks for the maximum number of unit distances that are possible among n points in the plane. Application of the Szemerédi-Trotter theorem provides $O(n^{4/3})$, and this is the best known estimate so far [29].

Dey [13] used the bounds known for the rectilinear crossing number for proving upper bounds on geometric k-sets. This led to considerable improvement on this bound after its early solution about 27 years ago.

The rectilinear crossing number of a complete graph is essentially the same as the minimum number of convex quadrilaterals determined by a set of n points in general position. It is known that the number of quadrilaterals is proportional to the fourth power of n, but the precise constant is unknown [28].

An important application of crossing numbers is in graph drawing and VLSI-design. This is why the author started research in this area in 1995. Figure 4 shows a graph with 120 objects and 161 edges that originated at an insurance company. The original drawing had 122 crossings, while the crossing minimal drawing has only 6 crossings.

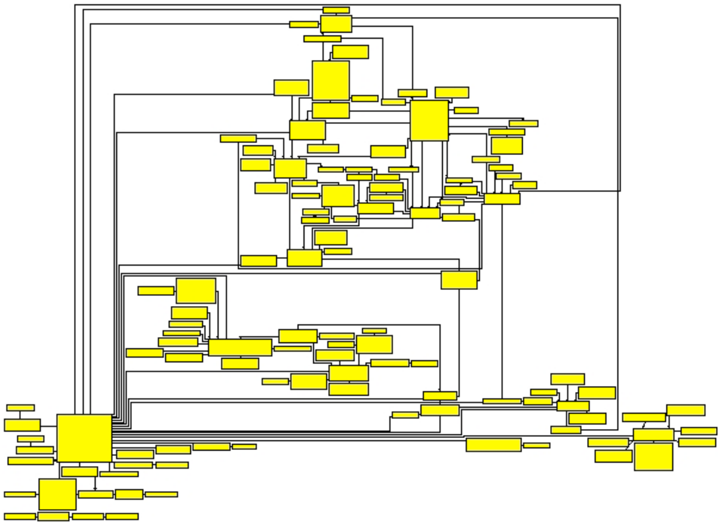

Fig. 4. A drawing of the insurance graph with 6 crossings

5 Approximation

Garey and Johnson have shown NP-completeness of the decision variant of the crossing number problem [21]. Pach and Tóth have shown the same for the crossing number variants *cr-odd* and *cr-pair*. Bienstock has shown that also the computation of $cr\text{-}lin(G)$ is NP-hard [3].

No polynomial time algorithm is known for approximating $cr(G)$ for general graphs within some non-trivial factor. Bhatt and Leighton [2] suggested the first algorithm which approximates $|V| + cr(G)$ for a bounded degree graph $G = (V, E)$ with a factor of $O(\log^4 |V|)$. This approximation factor has been improved by Even, Guha and Schieber [15] in 2002 to $O(\log^3 |V|)$. For sparse graphs, when $cr(G) = o(|V|)$, this approximation does not guarantee good results. Until recently, polynomial time algorithm approximating $cr(G)$ was known, not even for special graph classes.

Recently, the first approximation results have been achieved that do depend on the maximum degree and $cr(G)$ only. The approximation results concern the graph classes of *almost planar graphs* and *apex graphs*. A graph $G = (V, E)$ is called *almost planar* if G is non-planar, but there does exist an edge $e \in E$ so that $G - \{e\}$ is planar. Given a planar embedding Π of the remaining graph $G - \{e\}$, the edge e can be re-inserted with the minimal number of crossings via a shortest path in the extended geometric dual graph of Π. Gutwenger, Mutzel, and Weiskircher [16] have presented a linear time algorithm (based on the data structure of SPQR-trees) which is able to find the optimal embedding Π_0 of $G - \{e\}$, so that inserting e into Π_0 leads to a crossing minimum drawing over the set of all possible planar embeddings Π. The natural question arises, if this approach does approximate the crossing number $cr(G)$ by some small factor.

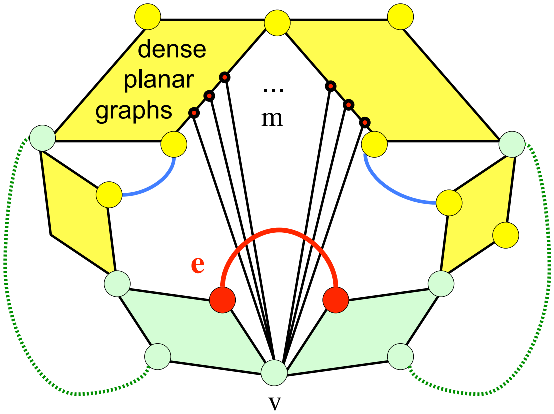

Fig. 5. Inserting edge e back into the planar graph $G - \{e\}$ yields m crossings if $\deg(v) \geq 2m$. Note that the shaded blocks are dense triconnected subgraphs. The crossing minimal drawing of G with 2 crossings can be found when flipping the two blocks adacent to v.

Figure 5 shows an example for which the edge e will be inserted into the planar graph $G - \{e\}$ with m crossings. However, the minimum crossing number of G is 2. This can be achieved by flipping the two lower components adjacent to v. Hliněný and Salazar [20] observed that the approximation factor in this case depends on the degree of vertex v. If the degree is bounded, then this example does not hurt the approximation anymore. Hliněný and Salazar have shown that the above algorithm provides crossing numbers of at most $\Delta(G - \{e\})cr(G)$, where $\Delta(G)$ denotes the maximum degree of G. This number has later been improved to $(\Delta(G - \{e\})/2)cr(G)$ by Cabello and Mohar [7]. This provides the first constant approximation algorithm for almost planar graphs with bounded degree graphs.

Very recently, these results could be generalized to apex-graphs. A graph $G = (V, E)$ is called an *apex graph* if G is non-planar, but there does exist a vertex $v \in V$ so that $G - \{v\}$ is planar. Chimani, Gutwenger, Mutzel, and Wolf [9] have shown that v and all its incident edges can be re-inserted into an optimal embedding Π_0 of $G - \{v\}$ (which the algorithm will identify) with the minimum number of crossings in polynomial time. Chimani, Hliněný, and Mutzel [10] have shown that this algorithm will find solutions which are at most a factor of $deg(v)\Delta(G - \{v\})/2$ away from the optimum solution $cr(G)$.

Both approximation results are (almost) tight: for almost planar graphs, there is an example showing that the approximation factor can be reached, while for apex graphs the example is still a factor of 2 away.

Some open questions arise:

- Is there a polynomial time algorithm for computing the crossing number $cr(G)$ for almost planar graphs? Cabello and Mohar [7] have shown that the weighted version is NP-hard.
- Can the above results be generalized, e.g., for graphs that are planar after deleting a fixed number of edges?

6 Exact Computation

Very few publications exist for computing the crossing number of general graphs exactly. Grohe has shown in 2001 that the crossing number problem is fixed-parameter tractable. However, the used concepts are based on the theoretical results of Robertson and Seymour. Recently, Kawarabayashi and Reed 2007 have improved the quadratic running time to a linear time algorithm for fixed k. Both approaches reduce the graph to one with bounded tree-width and the same crossing number and then test if the graph has crossing number at most k. There is common agreement that this approach is purely theoretical. Related to this, the following open problem is among the most important ones in the area: Can $cr(G)$ be computed in polynomial time for graphs with bounded tree-width?

Until 2005, no practically efficient algorithm for computing the crossing number was known. Today, there exist two approaches for computing the exact crossing number for general graphs. The approaches are based on two integer linear programming (ILP) formulations of the crossing number problem that can be solved by branch-and-cut algorithms.

The ILP formulation by Buchheim et al. [6] is called the *subdivision crossing minimization approach* (SOCM) and optimizes over the set of all simple drawings. A drawing is called *simple* if every edge is only crossed at most once. In order to provide an optimal solution for $cr(G)$, we need to subdivide all edges in G into a path of length $|E|$. The variables $x_{e,f}$ are associated with all non-adjacent edge pairs $(e, f) \in E^2$. The constraints come essentially from Kuratowski's theorem stating that a graph G is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5. Besides the Kuratowski-constraints and the 0/1-constraints, the ILP also contains constraints that guarantee to get simple drawings of the subdivided graph.

The second approach by Bomze, Chimani and Mutzel [11] is called the ordering-based ILP model (OOCM). This is not restricted to simple drawings. Instead, additional linear ordering variables y_{efg} are introduced for each edge $e \in E$ that may be crossed more than once. The variables y_{efg} for edges $e, f, g \in E$ provide the information in which order an edge e is crossed by f and g. In this ILP we need Kuratowski constraints on the x variables, linear ordering constraints on the y variables, 0/1-constraints for all variables, and additional linking constraints between the x and y variables.

We solve both ILP models with branch-and-cut algorithms. In order to get these algorithms to work in practice, we needed to come up with new preprocessing techniques as well as new combinatorial column generation methods. Our computational experiments on a benchmark set of about 11,000 graphs show that we can compute the exact crossing numbers for general sparse graphs with up to 100 vertices and crossing number up to 37 within 30 minutes.

Figure 6 shows the percentage of instances that have been solved within 30 minutes of computation time for about 11.000 graphs of the Rome library. The x-asis shows the number of vertices $|V|$. The number of edges of the Rome graphs is below $1.5|V|$ on average. The crossing number of almost all graphs with up to 60 vertices could be solved to provable optimality within 30 minutes CPU-time.

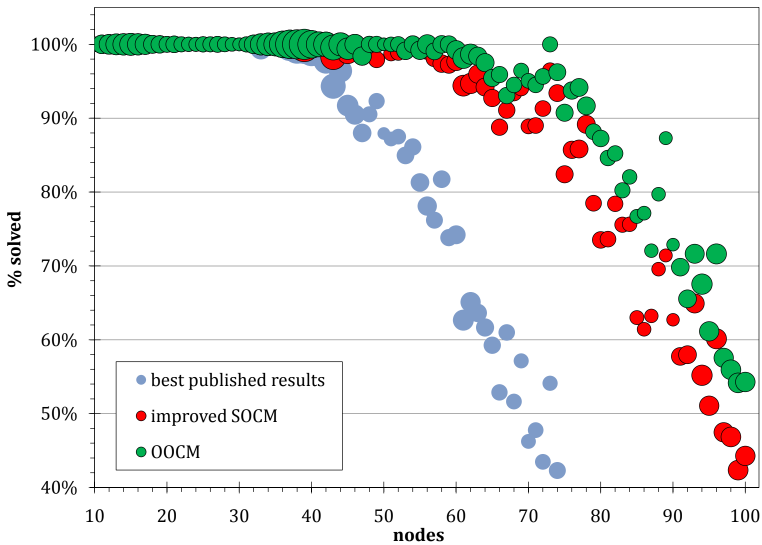

Fig. 6. The average percentage of instances solved within 30 minutes of computation time of the ILP models SOCM and OOCM

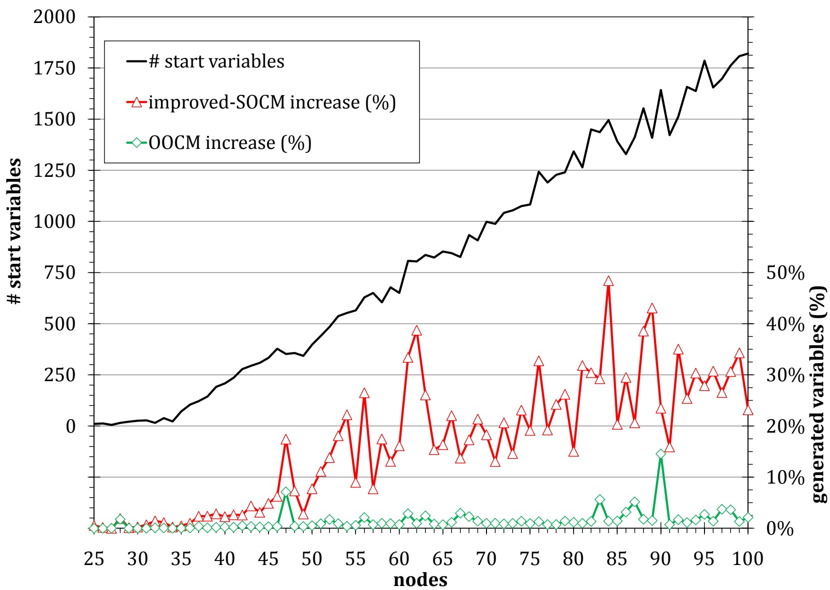

Fig. 7. The dark line shows the average number of start variables in our branch-and-cut approach (left axis). The right axis shows the percentage of additional variables generated by the two approaches.

The graphs with 100 vertices are much harder to solve. But still, more than 50% of the instances with 100 vertices could be solved to optimality.

It seems that the second formulation based on linear ordering dominates our first ILP model. For most instances, we need far more variables in our SOCM model than in the OOCM model. The dark line in Figure 7 shows the average number of start variables (left axis) in our branch-and-cut approach. For graphs with 100 vertices the average number was about 1800 variables. During the run of the algorithm, column generation adds in additional variables. These numbers are much higher for the SOCM approach. While SOCM added about 50% new variables with respect to the start variables, OOCM only had to add 18% additional variables for the 100-vertex graphs.

We find it surprising that in the OOCM model only very few y-variables are needed in order to find the optimum solution. Detailed experiments and results can be found in [8, 11].

7 Solved Open Problems

We close our survey with selected experimental results for special graph classes. While we could verify the crossing number of the complete graph on 11 (and 12) vertices (with an alternative optimum solution), one more vertex is still a challenge.

On the other hand, we are able to compute the crossing number of generalized Petersen graphs $P_{n,4}$ up to $n = 44$ which was unknown before. Based on our computed results, we came up with a conjecture of the crossing number of this graph class.

Moreover, we are confident that we are about to be able to compute the crossing number of the smallest toroidal grid graph $T_{8,8}$ whose crossing number is still unknown (and conjectured to be 48).

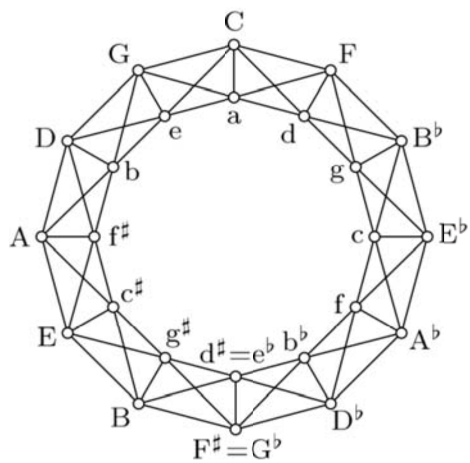

Fig. 8. Knuth's musical graph

In Exercise 133 of The Art of Computer Programming, Volume 4, Draft of Section 7, Donald E. Knuth uses the "musical graph" on page 73 of *Graphs* by R. J. Wilson and J. J. Watkins (1990) (see Fig. 8).

It represents simple modulations between key signatures. While all kinds of properties of this graph are easily analyzed, the question "Can it be drawn with fewer than 12 crossings?" remained open. After 9.71 seconds of computation time, our program proved that the crossing number is indeed 12 and produced an alternative embedding that is not as nice as the original, though.

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